Conservation law:

Firstly, consider the Burgers equation

$$\begin{cases} u_t + u u_x = 0 \\ u(0, x) = \phi(x). \end{cases}$$

The characteristic curve satisfies

$$\begin{cases} \frac{dx}{dt} = u(t, x) \\ x(0) = x_0 \end{cases}$$

Along this curve, $g(t) = u(t, x(t))$, $\dot{g}(t) = u_t + u_x \cdot x'(t) = 0$

$$\Rightarrow u(t, x(t)) = u(0, x_0) = \phi(x_0) = \phi(x - ut)$$

And the characteristic curve is $x = \phi(x_0) t + x_0$

It's naturally to notice that: if $\phi'(x) < 0$, the characteristic curve would intersect. This would cause dificulties in defining the solution beyond the intersection point. So we must sacrifice something, just like the "classical" property. to better define a solution. This means we need to allow some non-continous solutions.

Below, we list an example.

Consider a street starting at x_1, ending at x_2. The density of cars at point x, time t is $u(t, x)$.
Let f represent the flow rate onto and off the street.

$\to x_1 \qquad\qquad \to x_2.$

It's obviously that: $\frac{d}{dt}\int_{x_1}^{x_2} u(t,x)\,dx = f(u(x_1,t)) - f(u(x_2,t))$

$$\Rightarrow \frac{\int_{x_1}^{x_2} u_t(t,x)\,dx}{x_2-x_1} = \frac{f(u(x_1,t)) - f(u(x_2,t))}{x_2-x_1}$$

$x_2 \to x_1 \Rightarrow u_t = -[f(u)]_x$

Now, we begin to solve $\begin{cases} u_t + (f(u))_x = 0, \ x \in \mathbb{R}, t \geq 0 \\ u(0,x) = \varphi(x) \end{cases}$ (1)

We don't introduce the common weak solution's def, since it may use some advanced analysis knowledge. However, just like the weak derivative, we can use some test functions and integral by parts to relax the requirement that the solution need to satisfy.

More clearly, for all test functions $v \in C_c^\infty([0,\infty) \times \mathbb{R})$

If there exists a C^1 solution u, then

$$\int_0^\infty \int_{-\infty}^{+\infty} (u_t + (f(u))_x)\, v \,dx\,dt = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} \varphi(x) v(0,x)\,dx + \int_0^\infty \int_{-\infty}^{+\infty} u v_t + f(u) v_x \,dx\,dt = 0 \qquad (A)$$

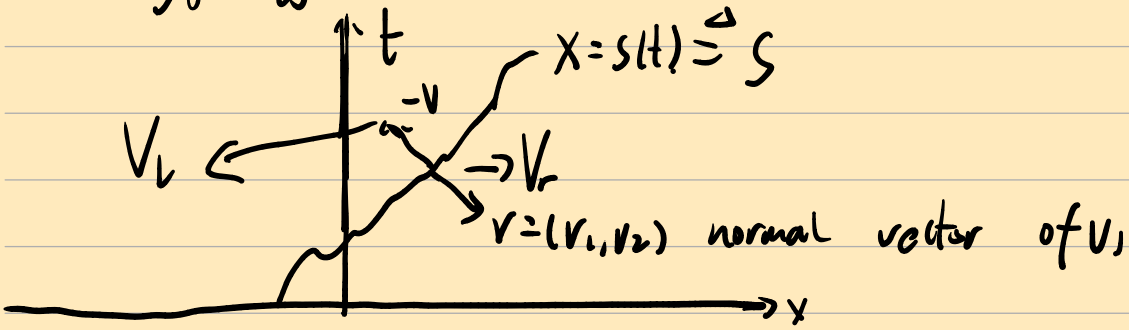
It's treasurable that a solution u of (A) may not be a classical solution of (1). And we say u is a weak solution of (1) if it satisfies (A) for all test $v \in C_c^\infty([0,\infty) \times \mathbb{R})$

From above, we can see that: classical solution is a weak solution. Similarly, we can show that: a C^1 weak solution is a classical solution.

Here, we discuss a simple condition, assuming that: the weak solution is discontinous across the curve $x=s(t)$, but is smooth (or C^1) on either side of $x=s(t)$.

Take test function $v \in C_c^\infty([0,\infty)\times\mathbb{R})$ with $v(0,x)=0$

$$\Rightarrow 0=\int_0^\infty\int_{-\infty}^{+\infty} u v_t + f(u) v_x \,dx\,dt$$



Consider $V_l = \{(t,x) \mid 0<t<\infty,\ -\infty<x<s(t)\}$

$V_r = \{(t,x) \mid 0<t<\infty,\ s(t)<x<+\infty\}$

$$\Rightarrow 0 = \iint_{V_l} + \iint_{V_r} (u v_t + f(u) v_x)\,dx\,dt$$

$$= -\left(\iint_{V_l} + \iint_{V_r}\right)(u_t v + (f(u))_x v)\,dx\,dt + \iint_S \overset{u(t,x) \text{ in } V_l}{u_l}\, v\cdot\nu_2 + f(u_l)\, v\cdot\nu_1\,ds$$

$$- \iint_S (u_r v\cdot\nu_2 + f(u_r)\, v\cdot\nu_1)$$

Since in V_l, V_r, the solution is classical, we have:

$$\iint_S (u_l-u_r)\, v\cdot\nu_2 + (f(u_l)-f(u_r))\, v\cdot\nu_1\,ds = 0$$

$\Rightarrow (U_l - U_r) V_2 + (f(U_l) - f(U_r)) V_1 = 0$

Using some geometric $\Rightarrow f(U_l) - f(U_r) = \dot{s}(t) \cdot (U_l - U_r)$

(This may be hard to understand, you can take a bdd open set V in $[0,\infty) \times \mathbb{R}$ and repeat this method. Then leverage that: V is arbitrary to get same result)

Define $\begin{cases} [U] = U_l - U_r \\ [f(U)] = f(U_l) - f(U_r) \end{cases}$) Jump of U and $f(U)$.
$\sigma = \dot{s}(t)$, speed of the curve.

$\Rightarrow [f(U)] = \sigma [U]$, so called Rankine-Hugoniot jump condition.

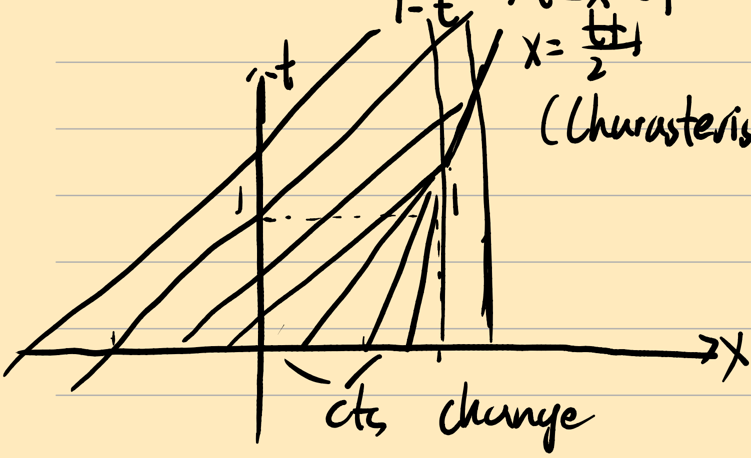
So why we consider this condition? Take a look at the following example.

$$\begin{cases} U_t + UU_x = 0, \ t \geq 0 \ \cong \ U_t + \left(\frac{U^2}{2}\right)_x = 0, \ t \geq 0 \\ U(0,x) = \phi(x) = \begin{cases} 1, & x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & x > 1. \end{cases} \end{cases}$$

The characteristic curve is $\phi(x_0) t + x_0$, along this $u(t,x)$ is a constant $\phi(x_0)$. $\Rightarrow x = \phi(x - ut)$

If $u(t,x) = U(0,x_0) = \phi(x_0) = 1 \Rightarrow x - t = x_0 < 0 \Rightarrow x < t$
If $U(t,x) = 0 \Rightarrow x = x_0 > 1$
If $u(t,x) = \phi(x_0) = 1 - x_0 \Rightarrow (1-x_0)t + x_0 = x \Rightarrow x_0 = \frac{x-t}{1-t} \in (0,1)$

$\Rightarrow u(t,x) = 1 - x_0 = \dfrac{1-x}{1-t}, \quad t < x < 1$



(Characteristic curve intersect)

From RH-condition. Notice that $f(u) = \frac{u^2}{2}$

The intersection point is $(1,1)$

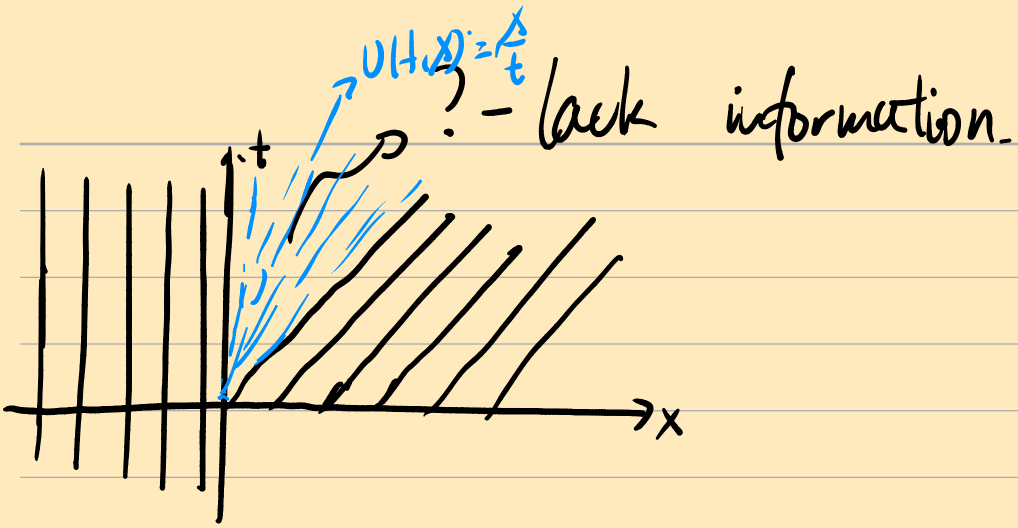
$$\dot{s}(t) = \frac{\frac{u_L^2 - u_r^2}{2}}{u_L - u_r} = \frac{\frac{1-0}{2}}{1-0} = \frac{1}{2} \Rightarrow s(t) = \frac{t}{2} + \frac{1}{2}, \quad t > 1.$$

If change $\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \Rightarrow u(t,x) = \begin{cases} 1, & x < \frac{t}{2} \\ 0, & x > \frac{t}{2} \end{cases}$

However, if we let $\phi(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$, it gives a different condition.

Still, the characteristic curve is $\phi(x_0)t + x_0$. Along this,
$u(t,x) = \phi(x_0) = \phi(x - ut) = u(0, x_0)$

Notice that: $\begin{cases} \text{if } x_0 < 0, \text{ characteristic curve is } x = x_0 \\ \text{if } x_0 > 0, \text{ characteristic curve is } x = x_0 + t \end{cases}$



Just using RH condition, we can figure out

$$u_1(t,x)=\begin{cases}0, & x<\frac{t}{2}\\ 1, & x>\frac{t}{2}\end{cases}$$ is a solution.

However $$u_2(t,x)=\begin{cases}0, & x\le 0\\ \frac{x}{t}, & 0\le x\le t\\ 1, & x\ge t\end{cases}$$ is also a solution, and is "cts"

(This is called rarefaction wave)

So among these solutions, which is more physically realistic?
To better understand this, we need to understand a concept — entropy condition.

Consider the quasilinear equation again.

$$u_t+[f(u)]_x=u_t+f'(u)u_x=0, \quad u(0,x)=\varphi$$

The characteristic curve is $\begin{cases}\frac{dx}{dt}=f'(u)\\ x(0)=x_0\end{cases}$ Along this curve, $u(t,x)=\varphi(x_0)$

For some reasons, we only allow the curve of discontinuity in our solution $u(t,x)$ if the wave to the left is moving faster than right, i.e. We may only allow for this curve if

$$f'(u_l)>\sigma>f'(u_r) \quad \text{(entropy condition).}$$

We say a curve of dis-cts is a shock curve for a solution u if the curve satisfies the RH condition and entropy condition for that solution u.

We say the solution u is a weak, admissible solution $\Leftrightarrow$ u is a weak solution, s.t any dis-cts curve for u is a shock curve.

We say f is uniformly convex (in optimization, strongly cvx) when $\exists\, \theta > 0$, s.t $f'' \geq \theta > 0$

If f is uniformly convex, then u is a weak, admissible solution $\Leftrightarrow U^- > U^+$ $(U_l > U_r)$

(此时，特征线为 $f'(\phi(x_0))t + x_0$)

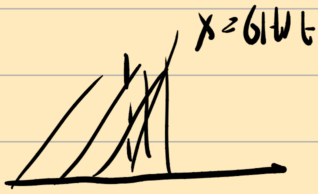
Here, we consider the Riemann's problem.

$$\begin{cases} U_t + [f(U)]_x = 0, & t \geq 0 \\ U(0,x) = \phi(x) = \begin{cases} U^-, & x < 0 \\ U^+, & x > 0 \end{cases} \end{cases}, \quad f'' \geq \theta > 0$$

Then there exists a unique weak admissible solution.

1. If $U^- > U^+$, then the admissible solution has a shock curve of speed σ and solution is.

$$U(t,x) = \begin{cases} U^-, & \frac{x}{t} < \sigma = \frac{[f(U)]}{[U]} \\ U^+, & \frac{x}{t} > \sigma \end{cases}$$



2. If $u^- < u^+$, then the solution has a rarefaction wave,

$$u(t,x) = \begin{cases} u^-, & \frac{x}{t} < f'(u^-) \\ (f')^{-1}(\frac{x}{t}), & f'(u^-) < \frac{x}{t} < f'(u^+) \\ u^+, & \frac{x}{t} > f'(u^+). \end{cases}$$

Proof: (首先验证 $\frac{[f(u)]}{[u]}$ 位于 $f'(u_r)$. $f'(u_l)$ 间，大于0？

1. $u(t,x)$ is a classical solution on either side of the curve

Check RH condition. Since $u^- > u^+$, $f'(u^-) > f'(u^+)$ --- 同

2. Let $a = (f')^{-1}$ (existence is a result of f''s monotone)

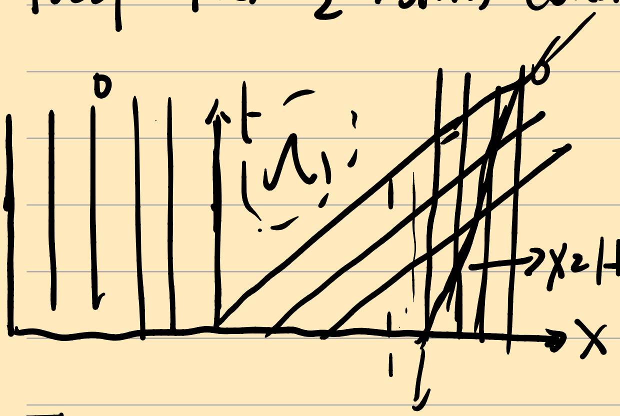
Check it's a solution in 3 areas and check the cts condition.

(uniqueness's pf is omitted here)

Notice that the Riemann problem can be extended to the periodic function - initial value.

For example. $\begin{cases} u_t + uu_x = 0, & t \geq 0 \\ u(0,x) = \phi(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise}. \end{cases} \end{cases}$

Proof: $f(u) = \frac{u^2}{2}$. still, consider $\begin{cases} \frac{dx}{dt} = (\frac{u^2}{2})' = u(t,x) = u(0,x_0) = \phi(x_0) \\ x(0) = x_0 \end{cases}$



in Ω_1, it's supposed to have:

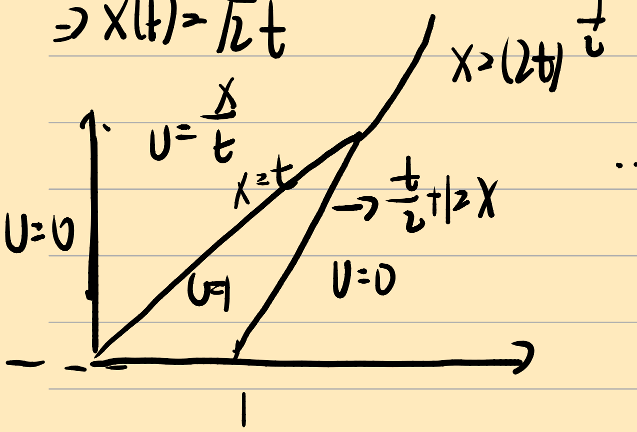
$u(t,x) = (f')^{-1}(\frac{x}{t}) = \frac{x}{t}$

The shock wave's velocity is $\frac{1}{2} \Rightarrow x = 1 + \frac{t}{2}$

And on the top, the rarefaction wave intersect with

Hence $\dot{\sigma}(t) = \dfrac{\frac{1}{2}\left(\frac{x}{t}\right)^2 - \frac{0^2}{2}}{\frac{x}{t}} = \dfrac{x}{2t} = \dot{x}(t)$, $x(2)=2$

$\Rightarrow x(t) = \sqrt{2t}$



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