Wave equation review.

$\Rightarrow \Box = \partial_{tt} - \Delta$

Consider the wave equation $\begin{cases} u_{tt} = \Delta u & \text{in } U \times [0,\infty) \overset{\Delta}{=} U_\infty \\ u(0,x) = g(x) & \text{(initial position)} \\ u_t(0,x) = h(x) & \text{(initial velocity)} \end{cases}$ in 1-dim

(u represents displacement at time t, location x).

I) Associated variation problem.

$L[u] = \int_{U_T} \frac{1}{2}|u_t|^2 - \frac{1}{2}|u_x|^2 \, dt\,dx$ defined for bdry condition ...

Consider for any $v \in C_0^\infty(U_\infty)$, let $i(\varepsilon) = L[u+\varepsilon v]$

$\Rightarrow i'(0) = \int_U u_t v_t - \nabla u \nabla v = \int_U (-u_{tt} + \Delta u) v = 0, \ \forall v \in C_0^\infty(U_\infty)$. ... $\square$

II) Two ways to derive to solution.

a. $\begin{cases} (u_{tt})^\wedge(t,\xi) = -4\pi^2|\xi|^2 \hat{u}(t,\xi) \\ \hat{u}(0,\xi) = \hat{g}(\xi) \\ \hat{u}_t(0,\xi) = \hat{h}(\xi) \end{cases} \Rightarrow \hat{u}(t,\xi) = \hat{g}(\xi)\cos 2\pi|\xi|t + \frac{\hat{h}(\xi)}{2\pi|\xi|}\sin 2\pi|\xi|t$

$\Rightarrow u(t,x) = g * (\cos 2\pi|\xi|t)^\vee + h * \left(\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}\right)^\vee$

$$\int_{-\infty}^{+\infty} e^{2\pi i \xi x} \cdot \cos 2\pi\xi t \, d\xi = \int_{-\infty}^{+\infty} \frac{e^{2\pi i\xi(x+t)} + e^{2\pi i \xi(x-t)}}{2} d\xi$$
$$= \frac{\delta(x+t) + \delta(x-t)}{2}$$

$$\Rightarrow u(t,x) = \frac{g(x+t) + g(x-t)}{2} + \frac{\int_{x-t}^{x+t} h(y)\,dy}{2}$$

$$\left(\left(\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}\right)^\vee = \int_0^t \frac{\delta(x+s) + \delta(x-s)}{2} ds = \frac{1}{2}\mathbb{1}_{(-t,t)} \right)$$

6. $(\partial_{tt}-\partial_{xx})=(\partial_t-\partial_x)(\partial_t+\partial_x)$

First, we solve $(\partial_t-\partial_x)U(t,x)=0$
Along the characteristic curve, $x(t)=x_0-t$, const $\Rightarrow U(t,x)=f(t+x)$

$\Rightarrow (\partial_t+\partial_x)U(t,x)=f(x+t)$
Along the characteristic curve $x(t)=x_0+t$, $\dot{U}(t,x(t))=f(x_0+2t)$

$\Rightarrow U(t,x)=\int_0^t f(x_0+2s)\,ds \overset{+U(0,x_0)}{\vee} = F(x+t)-F(x-t)+g(x-t)$

$\Rightarrow U(t,x)$ can be written in form of $f(x+t)+g(x-t)$

Substituting to initial condition ... ▯

Reflection principle.

$\begin{cases}(\partial_{tt}-\partial_{xx})U=0, & U=0 \text{ on } x=0.\\ U=g,\ U_t=h\end{cases}$ $x\geq 0$ We can extend u oddly.

$$\Rightarrow U(t,x)=\begin{cases}\dfrac{g(x+t)+g(x-t)}{2}+\dfrac{\int_{x-t}^{x+t}h(y)dy}{2}, & x\geq t\geq 0\\ \dfrac{g(x+t)-g(t-x)}{2}+\dfrac{\int_{t-x}^{x+t}h(y)dy}{2}, & t>x\geq 0\end{cases}$$

☆ Wave equation in higher dimensional spaces:

Consider $U(x,t,r)=⨍_{\partial B(x,r)} U(t,y)\,dS(y)$

$\begin{cases}G(x,r)=⨍_{\partial B(x,r)} g(y)\,dS(y)\\ H(x,r)=⨍_{\partial B(x,r)} h(y)\,dS(y)\end{cases}$

Fix $x \in \mathbb{R}^n$, we have:

$$U_r(x,t,r) = \left(⨍_{\partial B(0,1)} U(t, x+rz)\, dS(z) \right)_r$$
$$= ⨍_{\partial B(0,1)} DU(t, x+rz) \cdot z\, dS(z)$$
$$= ⨍_{\partial B(x,r)} DU(t,y) \frac{y-x}{r}\, dS(y)$$
$$= \frac{r}{n} ⨍_{B(x,r)} \Delta U(t,y)\, dy$$
$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta U(t,y)\, dy$$

$$U_{rr}(x,t,r) = \left(\frac{1-n}{n\alpha(n)} \frac{1}{r^n} \int_{B(x,r)} \Delta U(t,y)\, dy + \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \Delta U(t,y)\, dS(y) \right)$$

$$U_{tt}(x,t,r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} \Delta U(t,y)\, dS(y).$$

$$\Rightarrow U_{rr} + \frac{n-1}{r} U_r = U_{tt}.$$
$$\begin{cases} U(x,0,r) = G(x,r) \\ U_t(x,0,r) = H(x,r) \end{cases}$$

When $n=3$. $U_{rr} + \frac{2}{r} U_r = U_{tt}$, $\tilde{U}(x,r,t) = r\,U(x,r,t)$

$$\Rightarrow \tilde{U}_r = U + rU_r, \quad \tilde{U}_{rr} = 2U_r + rU_{rr} = \tilde{U}_{tt}$$

$$\Rightarrow \tilde{U}(x,t,r) = \frac{\tilde{G}(x,r+t) - \tilde{G}(x,t-r)}{2} + \frac{\int_{t-r}^{r+t} \tilde{H}(x,y)\, dy}{2} \quad \text{when } 0 \le r < t$$

$$\Rightarrow U(t,x) = \lim_{r \to 0^+} U(x,t,r) = \lim_{r \to 0^+} \frac{\tilde{U}(x,t,r)}{r} = \tilde{G}'(x,t) + \tilde{H}(x,t)$$

Compute $\tilde{G}'(x,t)$ and $\tilde{H}(x,t)$

$\tilde{G}(x,t) = t \fint_{\partial B(x,t)} g(y)\,dS(y) = \frac{1}{4\pi t}\int_{\partial B(x,t)} g(y)\,dS(y)$ → Important tech ?

$\tilde{G}'(x,t) = \fint_{\partial B(x,t)} g(y)\,dS(y) + t\left(\fint_{\partial B(0,1)} g(x+tz)\,dS(z)\right)'$

$= \fint_{\partial B(x,t)} g(y)\,dS(y) + t\fint_{\partial B(0,1)} Dg(x+tz)\cdot z\,dS(z)$

$= \fint_{\partial B(x,t)} g(y)\,dS(y) + t\fint_{\partial B(x,t)} Dg(y)\frac{y-x}{t}\,dS(y)$

$\Rightarrow u(t,x) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y)(y-x)\,dS(y)$

So how to derivate 2-dimensional solution?

Consider $u(t,x)$ solves WE in 2-dimensional space, we have:

$u(t,x) = \bar{u}(t,\bar{x})$ $\quad$ ($(x_1,x_2) \cong (x_1,x_2,0)$)

And define $\begin{cases} \bar{g}(x_1,x_2,x_3) = g(x_1,x_2) = g\circ\pi_{(x_1,x_2)}(x) \\ \bar{h}(x_1,x_2,x_3) = h\circ\pi_{(x_1,x_2)}(x) \end{cases}$ for $x\in\mathbb{R}^3$

projection from $\mathbb{R}^3$ to x_1-x_2 plane

So $\bar{u}(t,\bar{x}) = \fint_{\partial B(\bar{x},t)} th(y) + g(y) + Dg(y)(y-\bar{x})\,dS(y)$

Consider the mapping $y\in\mathbb{R}^2 \to (y, \sqrt{t^2-|y-x|^2}) \triangleq f(y)$.

$\Rightarrow \bar{u}(t,\bar{x}) = \frac{1}{2}\fint_{B(x,t)} th(y) + g(y) + Dg(y)(y-x)\cdot\sqrt{1+|\nabla f(y)|^2}\,dy$

$= \frac{1}{2}\fint_{B(x,t)} \frac{t^2h(y) + tg(y) + tDg(y)(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy$

Here comes the question:

How to solve $\begin{cases} U_{tt} = \Delta U + f \\ U(0,x) = 0 \\ U_t(0,x) = 0 \end{cases}$

Let $v(t,x,s)$ solves $\begin{cases} V_{tt} = V_{xx}. \\ V(s,x,s) = 0, \ V_t(s,x,s) = f(s,x) \end{cases}$ for each fixed s.

Consider $U(t,x) = \int_0^t v(t,x,s)\,ds$.

Then $\begin{cases} U_t(t,x) = v(t,x,t) + \int_0^t v_t(t,x,s)\,ds = \int_0^t v_t(t,x,s)\,ds \\ U_{tt}(t,x) = V_t(t,x,t) + \int_0^t V_{tt}(t,x,s)\,ds = f(t,x) + U_{xx} \end{cases}$

So for

① $\mathbb{R}^3$, we have: $v(t,x,s) = ⨍_{\partial B(x,t-s)} f(s,y)\,(t-s)\,dy$

② $\mathbb{R}$, we have, $v(t,x,s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy$.

So far, you may ask. aha, what you just wrote is "the speed of the wave is 1". so how to deal with.

$\begin{cases} U_{tt} = a^2 \Delta U + f \\ \sim . \end{cases}$ just let $v(t,x) = u(t,ax)$

$\Rightarrow V_{tt}(t,x) = U_{tt}(t,ax) = a^2 \Delta u(t,ax) \overset{+f(t,ax)}{=} \Delta v(t,x) + f(t,ax) \ldots$

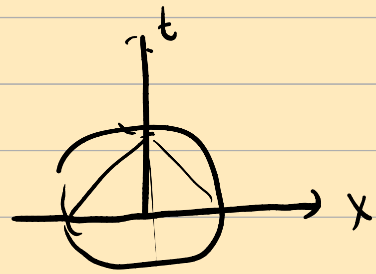
Now, we propose the energy estimate to prove that: the classical solution of the wave equation is unique.

The solution for $\begin{cases} u_{tt} = \Delta u + f \\ u(0,x) = g \\ u_t(0,x) = h \end{cases}$ in $\mathbb{R}^n$ is unique.
$C^2(U_\infty)$

Proof: Only need to show, the C^2 solution for $f=g=h=0$ is only 0.

Consider the whole energy $e(t) = \frac{1}{2}\int_U |u_t|^2 + |\nabla u|^2 \, dx$!!!

$$\begin{aligned} \dot{e}(t) &= \int_U u_t u_{tt} + \nabla u \nabla u_t \, dx \\ &= \int_U u_t \Delta u + \nabla u \nabla u_t \, dx \\ &= \int_{\partial U} \frac{\partial u_t}{\partial n} \nabla u \, dx = 0 \quad \text{with } e(0) = 0 \end{aligned}$$



Domain of dependence

Consider $\Box u = 0$, and if $u = u_t = 0$ in $\{t=0\} \times B_{t_0}(x_0)$, $u \in C^2 \Rightarrow u \equiv 0$ in the cone $\{(t,x) \mid 0 \le t \le t_0, |x - x_0| \le t_0 - t\}$

Proof: Consider the energy again, we have:

Let $e(t) = \frac{1}{2}\int_{B_{t_0-t}(x_0)} |\nabla u|^2 + |u_t|^2 \, dx = \frac{1}{2}\int_0^{t_0-t}\int_{\partial B(x_0,s)} |\nabla u|^2 + |u_t|^2 \, dS \, ds$

(A) $$\begin{aligned} \dot{e}(t) &= \int_{B_{t_0-t}(x_0)} \nabla u \cdot \nabla u_t + u_t \cdot \underbrace{u_{tt}}_{\Delta u} \, dx \\ &\quad - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} |\nabla u|^2 + |u_t|^2 \, dS \\ &= \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial n} u_t \, dS - \quad \le 0 \end{aligned}$$

You may ask, what happens when $\Box u = f$.

Still consider $e(t) = \frac{1}{2}\int_{B_{t_0-t}(x_0)} |u_t|^2 + |\nabla u|^2 dx$

$$\dot{e}(t) = \int_{B_{t_0-t}(x_0)} u_t u_{tt} + \nabla u \nabla u_t \, dx - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} |u_t|^2 + |\nabla u|^2 dx$$

$$\leq \int_{B_{t_0-t}(x_0)} u_t f \leq e(t) + \frac{1}{2}\int_{B_{t_0-t}(x_0)} f^2 dx$$

Use Grownwall $\square$

Before we end our Review, we propose the separation of variables for wave function in 1-dimension. (i.e. in $(0,l)$)

$$\begin{cases} u_{tt} = \Delta u + f \\ u(0,x) = g(x) \\ u_t(0,x) = h(x) \\ u(t,0) = g_1(t) \\ u(t,l) = g_2(t) \end{cases}$$

Still, consider what we do in HZ.

$$u(t,x) - \begin{cases} \frac{l-x}{l} g_1(t) \\ \frac{(l-x)^2}{-2l} g_1(t) \end{cases} - \begin{cases} \frac{x}{l} g_2(t) \\ \frac{x^2}{2l} g_2(t) \end{cases} \text{ to let } g_1 = g_2 = 0$$

$$u_x(t,0) = g_1(t) \qquad u_x(t,l) = g_2(t)$$

Consider $X_n(x) T_n(t)$ which solves the equation.

$$\frac{X_n''(x)}{X_n(x)} = \frac{T_n''(t)}{T_n(t)} = -\lambda, \ X_n(0) = X_n(l) = 0 \ (\text{S-L problem})$$

$$\Rightarrow \begin{cases} X_n''(x) + \lambda X_n(x) = 0 \\ X_n(0) = X_n(l) = 0 \end{cases} \Rightarrow X_n(x) = c\sin\sqrt{\lambda}x, \ \sqrt{\lambda}l = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$$

And $X_n(x)$ forms an orthogonal basis in $L^2(0,l)$

Let $$\begin{cases} f(x) = \sum_{n=1}^{\infty} f_n(t) X_n(x) \\ g(x) = \sum_{n=1}^{\infty} g_n X_n(x) \\ h(x) = \sum_{n=1}^{\infty} h_n X_n(x) \end{cases}$$

$$\Rightarrow \begin{cases} T_n''(t) + \lambda T_n(t) = f_n(t) \\ T_n(0) = g_n \\ T_n'(0) = h_n \end{cases}$$

(If you do odd extension, it corresponding with the wave solution's form above).

A special case for $f(t,x)$ is $f_n = A_n \sin \omega t$, let $L = \pi$.

$$\Rightarrow T_n''(t) + n^2 T_n(t) = A_n \sin \omega t.$$

From Duhamel's principle.

$$T_n(t) = \int_0^t \left(\frac{\sin n s}{n}\right) \cdot A_n \sin \omega (t-s)\, ds$$

$$= \int_0^t \frac{\sin n s}{n} A_n \sin \omega (t-s)\, ds + \text{homogeneous solution}$$

$\Rightarrow \|u\|_{L^\infty} \to \infty$ if $|\omega| = n$. / No maximum principle exists.